

## FLEXURE OF BEAMS RESTING ON HYPERBOLIC ELASTIC FOUNDATIONS

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**Abstract**—This article examines the static problem of the flexure of a Bernoulli-Euler beam resting on a nonlinear Winkler-type foundation. A perturbation technique is used to solve the nonlinear differential equation associated with the problem. Using this technique, the initially nonlinear problem is reduced to the solution of a set of linearised equations. For the successive solution of these equations, some analytical methods are outlined. These methods are applicable to either finite or infinite beams. As an example of the applications of the proposed analysis, the problem of the flexure of a finite beam subjected to a concentrated line load, applied at an arbitrary point of the beam, is solved. In a second example, the solution of the problem of the flexure of a finite beam having free edges and subjected to an initial displacement at its middle point is presented.

### 1. INTRODUCTION

The analysis of finite or infinite beams resting on linearly elastic deformable media has received considerable attention. An extensive review of the subject is given in [1].

The present study is concerned with the solution of the problem of flexure of a linearly elastic beam resting on a nonlinear Winkler-type foundation. It is assumed that the beam, which may be of finite or infinite length, is subjected to an arbitrarily distributed transverse load. The nonlinear elastic foundation is a hyperbolic one; that is, a hyperbolic-type nonlinear relation is adopted to relate the stress, applied at any point of the foundation surface, to the corresponding deflection.

For the governing differential equation of the problem, which is essentially nonlinear, an asymptotic expansion solution is assumed and a perturbation solution technique is employed. Hence, the original problem reduces to the solution of a set of linearised differential equations. Each one of these equations may be considered as the equation that describes the bending problem of a beam resting on a linear elastic Winkler foundation and subjected to a certain type of external loading.

For the successive solution of the linearised equations, some analytical methods are outlined for finite or infinite beams subjected to any type of external loading as well as any set of homogeneous or inhomogeneous boundary conditions. In more detail, in any case in which the beam is of finite length, for each one of these linearised equations, a generalised Fourier series solution may be obtained by employing the method of Galerkin[2]. As an alternative technique, a method based on the Laplace transformation, also suitable in any case in which the beam is a semiinfinite one, is outlined. Finally, a solution in terms of Fourier integrals may be obtained in any case in which the beam is an infinite one (extended from minus to plus infinity).

As an example of the applications of the proposed analysis, two particular problems concerning the flexure of a finite beam resting on a Winkler foundation of the hyperbolic type are solved. In the first problem, the beam is subjected to a concentrated line load. The load is applied, normally, at an arbitrary point of the beam. The beam is subjected to any set of homogeneous edge boundary conditions. In the second problem, there is no external loading applied on the beam. The beam is supposed to be subjected to a particular set of inhomogeneous boundary conditions chosen so that the obtained solution is also applicable to the problem of the flexure of a free-free beam subjected, at its middle point, to an initial displacement.

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Finally, as a numerical example, some numerical results are presented and discussed for the problem of the flexure of a finite beam having free edges and subjected to a concentrated line load applied at its middle point. A comparison is also made with some corresponding results obtained from the solution of the problem of the flexure of a beam having also free edges but subjected to an initial displacement at its middle point.

## 2. PROBLEM FORMULATION

In Fig. 1, the nomenclature of a beam resting on an elastic foundation is shown. It is assumed that the beam, which may be of finite or infinite length, is a slender one so that its small deflection flexural behaviour is governed by the classical Bernoulli-Euler beam theory. It is further assumed that the beam is subjected to an external transverse stress distribution  $p(x)$  and that the contact between the beam and the foundation is smooth and bilateral.

The foundation is assumed to be of the Winkler type in the sense that the displacement occurs immediately under the loaded area and outside this region the displacements are zero[1]. For the problem under consideration, a hyperbolic-type nonlinear relation of the form

$$q(x) = \frac{kw(x)}{1 + \mu w(x)} \quad (1)$$

is adopted to relate the stress  $q$  applied at any point  $x$  of the foundation surface to the corresponding deflection  $w$ . In eqn (1),  $k$  is the modulus of the linear subgrade reaction, with dimensions of stress per unit length, and  $\mu$  is a parameter, with dimensions of  $(\text{length})^{-1}$ , which can be considered a measure of the nonlinear response of the elastic foundation. When  $\mu = 0$ , eqn (1) describes the response of a linear elastic Winkler foundation.

Under these circumstances, the Bernoulli-Euler equation for the beam flexural displacement  $w$  is given as follows:

$$EI \frac{d^4 w}{dx^4} + \frac{kbw}{1 + \mu w} = bp(x), \quad (2)$$

where the Young's modulus  $E$  of the beam material, the moment of inertia  $I$  and the width  $b$  of the beam have been considered as constants.

Because of the nonlinear character of eqn (2), it is natural to enquire about the stability of any possible solution. Thus, according to the "principle of the minimum potential energy"[3], the positive definiteness of the second variation of the potential energy of the beam foundation elastic system must be examined. The strain energy  $V$

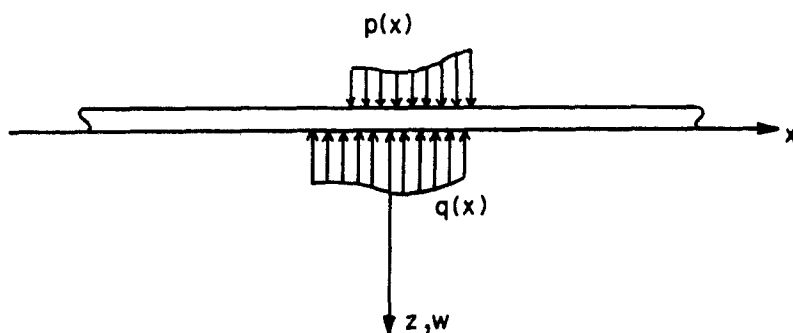


Fig. 1. Nomenclature of a beam resting on an elastic foundation.

is given by

$$V = U_b + U_f - W, \tag{3}$$

where

$$U_b = \frac{EI}{2} \int_{l_1}^{l_2} (w'')^2 dx, \tag{4a}$$

$$U_f = \frac{kb}{\mu^2} \int_{l_1}^{l_2} [1 + \mu w - \log_e (1 + \mu w)] dx \tag{4b}$$

and

$$W = b \int_{l_1}^{l_2} p(x) w dx, \tag{4c}$$

which are the strain energy of the beam, the strain energy of the hyperbolic foundation and the work done by the external stress distribution, respectively. The set of values of  $(l_1, l_2)$  determines the edges and the length of the beam. Thus,  $(l_1, l_2) = (0, l)$  for a finite beam of length  $l$ ,  $(l_1, l_2) = (0, \infty)$  for a semiinfinite beam or  $(l_1, l_2) = (-\infty, \infty)$  for an infinite beam.

For equilibrium, the vanishing of the first variation  $\delta V$  of the potential energy functional yields the governing equation of the problem (2), as well as all possible sets of the associated homogeneous boundary conditions. Each one of these sets contains two of the following homogeneous boundary conditions,

$$(i) \text{ at a clamped edge: } w = w' = 0, \tag{5a}$$

$$(ii) \text{ at a pinned edge: } w = w'' = 0, \tag{5b}$$

$$(iii) \text{ at a sliding edge: } w' = w'' = 0 \tag{5c}$$

and

$$(iv) \text{ at free edge: } w'' = w''' = 0. \tag{5d}$$

Hence, as may be apparent, each one of the sets of homogeneous edge boundary conditions associated with (2) coincides with one of the sets of all possible homogeneous boundary conditions associated with the corresponding linear bending problem of the beam (see, for instance, [4]). This is because the nonlinearity enters the problem only through the response of the elastic foundation, which has been expressed in terms of  $w(x)$  [derivatives of  $w$  do not appear in (1)].

However, a finite or a semiinfinite beam may also be subjected to inhomogeneous boundary conditions. In what follows, any set of edge boundary conditions in which at least one is inhomogeneous will be referred to as a set of inhomogeneous boundary conditions.

The second variation of the potential energy can be obtained in the form

$$\delta^2 V = \int_{l_1}^{l_2} \left[ EI(\delta w'')^2 + kb \frac{(\delta w)^2}{(1 + \mu w)^2} \right] dx, \tag{6}$$

which, because of the appearance of the squared terms, is always positive definite. Therefore, any possible solution of eqn (2) is a stable one.

### 3. METHOD OF SOLUTION

The governing equation of the problem, (2), may also be written in the following nondimensional form:

$$\lambda \frac{d^4 w}{d\xi^4} + \frac{4\lambda w}{1 + \mu w} = \frac{4\lambda}{k} p(\xi), \quad \xi = \lambda x, \quad (7)$$

where  $\lambda^{-1} = (kb/4EI)^{-1/4}$ , with dimensions of length, is the parameter that, in the case of a linear elastic Winkler foundation ( $\mu = 0$ ), is termed as the characteristic length of the beam foundation system. Hence,  $\xi \in (\lambda l_1, \lambda l_2)$  is a nondimensional spatial coordinate.

We assume that

$$|\mu w(\xi)| \ll 1, \quad \xi \in (\lambda l_1, \lambda l_2) \quad (8)$$

either because of a small values of  $\mu$  or, in the region of validity of the small deflection theory used here, because of small values of  $w(\xi)$ . Hence, for analysis of the class of problems for which (8) holds, we use a perturbation scheme. To this end, it is assumed that an auxiliary length parameter  $c$  exists such that  $\epsilon = \mu c$  is a small number. As will be apparent later, both parameters  $c$  and  $\epsilon$  can arise naturally in the formulation of a particular problem in which the length of the beam, the type of the external loading and the set of the edge boundary conditions will be specified. Under these circumstances, eqn (7) may be rewritten as follows:

$$(c + \epsilon w) w^{(IV)} + 4cw = \frac{4}{k} (c + \epsilon w) p(\xi), \quad (9)$$

where  $w^{(IV)} = d^4 w/d\xi^4$ .

Using  $\epsilon$  as a perturbation parameter, we propose, for eqn (9), an asymptotic expansion solution of the form

$$w(\xi) = \sum_{n=0}^{\infty} w_n(\xi) \epsilon^n, \quad (10)$$

where it has been assumed that the sequence  $w_n(\xi)\epsilon^n$  satisfies all necessary conditions that guarantee the uniform convergence of this series for  $\xi \in (\lambda l_1, \lambda l_2)$ . It is further assumed that each one of the unknown functions  $w_n(\xi)$  ( $n = 0, 1, \dots$ ) satisfies certain boundary conditions chosen in such a way that the whole series, that is,  $w(\xi)$ , satisfies the boundary conditions of eqn (7). Substituting the expansion (10) into (9) and equating coefficients of like powers of  $\epsilon$ , we obtain

$$w_0^{(IV)} + 4w_0 = \frac{4}{k} p(\xi), \quad n = 0 \quad (11a)$$

$$w_n^{(IV)} + 4w_n = \frac{4}{ck} p(\xi) w_{n-1} - \frac{1}{c} \sum_{i=0}^{n-1} w_i w_{n-i-1}^{(IV)}, \quad n = 1, 2, \dots \quad (11b)$$

The result [eqn (11a)] is the differential equation that describes the problem of the flexure of a beam resting on a linear elastic Winkler foundation and subjected to an external stress distribution  $p(\xi)$ ; for its solution, several methods have been developed for finite or infinite beams. Furthermore, each one of (11b) ( $n = 1, 2, \dots$ ) may be considered as the differential equation that describes the same problem, provided that the right-hand side is considered as a function describing an external stress distribution that is expressed in terms of  $p(\xi)$  as well as all previous  $w_i(\xi)$  ( $i = 0, 1, \dots, n-1$ ). Hence, using a solution of (11a) as a starting point, the solution of each one of (11b) may successively be obtained.

Because of the linear character of each one of (11), and since  $c^{-1} = \mu/\epsilon$ , (11) may

also be obtained in the following alternative form:

$$w_0^{(IV)} + 4w_0 = 4p(\xi), \quad n = 0 \tag{12a}$$

$$w_n^{(IV)} + 4w_n = 4p(\xi) w_{n-1} - \sum_{i=0}^{n-1} w_i w_{n-i}^{(IV)}, \quad n = 1, 2, \dots \tag{12b}$$

provided that the asymptotic series (10) will be replaced as follows:

$$w(\xi) = k^{-1} \sum_{n=0}^{\infty} \left(\frac{\mu}{k}\right)^n w_n(\xi). \tag{13}$$

Obviously, the auxiliary length parameter  $c$  as well as the perturbation parameter  $\epsilon$  do not appear explicitly in the solution of the problem. However, as will later be apparent, in some examples, both parameters arise naturally in any particular problem; they may be determined whenever the solution of (12),  $w_n(\xi)$  ( $n = 0, 1, \dots$ ), may be obtained.

In the remainder of this section, some analytical techniques will be outlined for the successive solution of the differential equations (12). The applicability of each of these techniques to particular problems is dependent on the length of the beam (finite, semiinfinite or infinite) but essentially independent of both the analytical form of the external stress distribution  $p(\xi)$  and the set of boundary conditions prescribed at the ends of the beam.

However, even though with one of these techniques the solution  $w_n(\xi)$  of each of (12) may be obtained, for the evaluation of the flexural displacement of the beam  $w(\xi_0)$  at a particular point  $\xi_0 \in (\lambda_1, \lambda_2)$ , an infinite number of terms must be summed in (13). Apparently, in a numerical application, the series of (13) must be truncated to a number of terms (say,  $N$ ) chosen so that for the obtained numerical results, convergence can be ensured to a desired accuracy.

### 3.1 The finite beam

An extensive review of the numerous analytical methods and approximate techniques developed for the analysis of finite beams resting on a linear Winkler foundation is given in [1]. It seems that many of them may be used to provide (11) with a recurrence-type solution, especially when both the external stress distribution  $p(\xi)$  and the boundary conditions are given.

The problem of the flexure of a finite beam of length  $l$  subjected to any kind of external loading as well as any set of homogeneous edge boundary conditions was studied by Iyengar and Anantharamu[5]. They obtained a solution in the form of a series of characteristic eigenfunctions of a freely vibrating beam. Here, a similar technique, in combination with the method of Galerkin[6], will be employed, and a solution will be presented for the problem of the flexure of a finite beam subjected to any set of homogeneous or inhomogeneous boundary conditions.

Let  $\beta = \lambda l$  be a nondimensional parameter that denotes the relative length of the beam. Also let

$$X_m \left(\frac{\lambda_m x}{l}\right) = X_m \left(\frac{\lambda_m \xi}{\beta}\right), \quad m = 1, 2, \dots \tag{14}$$

be the set of the characteristic functions that describe the normal modes of vibration of a beam subjected to a certain type of homogeneous boundary condition, where  $\lambda_m$  values are the roots of the corresponding frequency equation (see, for instance, [7-9]). In the interval  $(0, \beta)$ , the functions  $X_m$  satisfy the following orthogonality conditions:

$$\int_0^\beta X_m \left(\frac{\lambda_m \xi}{\beta}\right) X_r \left(\frac{\lambda_r \xi}{\beta}\right) d\xi = \beta \|X_m\|^2 \delta_{mr}, \quad m, r = 1, 2, \dots \tag{15a}$$

where  $\delta_{mr}$  is Kronecker's delta and  $\|X_m\|^2$  is the square norm of  $X_m$ , in  $(0, l)$ , defined according to

$$\|X_m\|^2 = \int_0^l [X_m(\lambda_m \eta)]^2 d\eta, \quad m = 1, 2, \dots \quad (15b)$$

This is equal to  $\frac{1}{2}$ , for a beam whose each edge is either a pinned or a sliding one, or to 1, for a beam with different end conditions. Also,

$$X_m^{(IV)} \left( \frac{\lambda_m \xi}{\beta} \right) = \left( \frac{\lambda_m \xi}{\beta} \right)^4 X_m \left( \frac{\lambda_m \xi}{\beta} \right), \quad m = 1, 2, \dots \quad (16)$$

Under the conditions of (15), the functions

$$\Phi_m \left( \frac{\lambda_m \xi}{\beta} \right) = X_m \left( \frac{\lambda_m \xi}{\beta} \right) / \sqrt{\beta} \|X_m\|, \quad m = 1, 2, \dots \quad (17)$$

form an orthonormal set of functions[10], in  $(0, \beta)$ , which is also complete, except for any one of the particular cases of a free-free, a free-pinned, a free-sliding or a sliding-sliding beam. However, in each one of these particular cases, the orthonormal set of the functions  $\Phi_m$  can be constructed to be complete too. This can be done by adding to the formal set of the functions  $X_m$  some suitably chosen functions (see Appendix and [9]). Furthermore, (16) is also valid for each one of these additional functions.

Let us now suppose that the finite beam is subjected to a certain set of inhomogeneous boundary conditions. We require from the solution of (12a)  $w_0(\xi)$  to satisfy this set of boundary conditions, whereas from the solution of each one of (12b)  $w_n(\xi)$  ( $n = 1, 2, \dots$ ) to satisfy the set of the corresponding homogeneous ones. These requirements guarantee that the asymptotic expansion solution (13) will satisfy the set of the inhomogeneous boundary conditions of the problem.

The general solution of (12a) may be written as

$$w_0(\xi) = u_0(\xi) + v_0(\xi), \quad (18)$$

where  $u_0(\xi)$  is a particular solution of (12a), whereas

$$v_0(\xi) = e^{\xi}[C_1 \cos(\xi) + C_2 \sin(\xi)] + e^{-\xi}[C_3 \cos(\xi) + C_4 \sin(\xi)], \quad (19)$$

is the general solution of the corresponding homogeneous differential equation. The constant coefficients  $C_1, \dots, C_4$  must be determined so that the set of the inhomogeneous boundary conditions of the problem are satisfied. Consequently, the particular solution  $u_0(\xi)$ , like each one of  $w_n(\xi)$  ( $n = 1, 2, \dots$ ), must satisfy the corresponding homogeneous boundary conditions. Hence, the unknown functions  $u_0(\xi)$  ( $n = 1, 2, \dots$ ) as well as the external stress distribution  $p(\xi)$  may be expressed in the following generalised Fourier series forms:

$$[u_0(\xi), w_n(\xi), p(\xi)] = \sum_{m=1}^{\infty} [A_{0m}, A_{nm}, p_m] X_m \left( \frac{\lambda_m \xi}{\beta} \right), \quad n = 1, 2, \dots \quad (20)$$

where  $X_m$  represents the set of the characteristic beam functions that satisfy the homogeneous boundary conditions

$$p_m = \frac{1}{\beta \|X_m\|^2} \int_0^{\beta} p(\eta) X_m \left( \frac{\lambda_m \eta}{\beta} \right) d\eta, \quad m = 1, 2, \dots \quad (21)$$

and  $A_{nm}$  ( $n = 0, 1, 2, \dots$ ) are unknown constant coefficients; they will be determined by applying, successfully, on (12) the method of Galerkin. Thus, taking into account (15) and (16), we obtain

$$A_{0m} = \frac{p_m}{1 + (\lambda_m/\sqrt{2}\beta)^4}, \quad n = 0. \quad (22a)$$

$$A_{nm} = \frac{1}{\beta \|X_m\|^2 [1 + (\lambda_m/\sqrt{2}\beta)^4]} \int_0^\beta \left\{ p(\xi) w_{n-1} - \frac{1}{4} \sum_{i=0}^{n-1} w_i w_{n-i-1}^{(IV)} \right\} X_m \left( \frac{\lambda_m \xi}{\beta} \right) d\xi, \quad n > 0. \quad (22b)$$

Clearly, (22) constitutes a recurrence-type formula. Through this formula, each one of the unknown constant coefficients  $A_{nm}$  may recurrently be evaluated, provided that the external stress distribution  $p(\xi)$  is given in an analytical form so that the coefficients  $p_m$  may be evaluated through (21), either analytically or numerically. Then, through (18)–(20), each one of  $w_n$  ( $n = 0, 1, 2, \dots$ ) may be evaluated at any particular point  $\xi_0 \in (0, \beta)$ , and hence the deflection  $w(\xi_0)$  of the beam may be obtained through the asymptotic series (13).

3.1.1 *Finite beam subjected to homogeneous boundary conditions.* As a particular case of the presented analysis, we consider here the bending problem of a finite beam subjected to a set of homogeneous edge boundary conditions. In this case, it can be easily shown that the constant coefficients  $C_1, \dots, C_4$  appearing in (19) are identically equal to zero, so that  $v_0(\xi) = 0$  and consequently

$$w_0(\xi) \equiv u_0(\xi) = \sum_{m=1}^{\infty} A_{0m} X_m \left( \frac{\lambda_m \xi}{\beta} \right), \quad (23)$$

where  $X_m$  is the set of the corresponding, to the particular boundary conditions, beam functions and the Fourier coefficients  $A_{0m}$  are given by (22a). Furthermore, the coefficients  $A_{nm}$  ( $n = 1, 2, \dots$ ) given by (22b) may be obtained in the form

$$A_{nm} = \frac{1}{\|X_m\|^2 [1 + (\lambda_m/\sqrt{2}\beta)^4]} \sum_{r=1}^{\infty} \sum_{l=1}^{\infty} \left[ A_{(n-1)r} p_l - \left( \frac{\lambda_l}{\sqrt{2}\beta} \right)^4 \sum_{i=0}^{\infty} A_{ir} A_{(n-i-1)l} \right] I_{mrl}, \quad n = 1, 2, \dots \quad (24)$$

where

$$I_{mrl} = \int_0^1 X_m(\lambda_m \eta) X_r(\lambda_r \eta) X_l(\lambda_l \eta) d\eta. \quad (25)$$

Although from (22a) the exact values of  $A_{0m}$  may be obtained in terms of the values of the corresponding  $p_m$  and  $\lambda_m$  values, for the evaluation of each one of  $A_{nm}$  ( $n = 1, 2, \dots$ ), a series of a double infinite number of terms must be summed in the recurrence formula (24). Furthermore, for the evaluation of each one of  $w_n(\xi_0)$  ( $n = 0, 1, 2, \dots$ ) at a particular point  $\xi_0 \in (0, \beta)$ , an infinite number of terms must be summed in (20). In a numerical application, all these series must be truncated. The most convenient technique is to truncate each one of the series (20) to a number of terms (say,  $M$ ) chosen so that convergence is ensured to a desired accuracy.

A further observation of the recurrence formulas (22) and (24) indicates that  $M$  depends on  $\beta = \lambda l$ . Since  $\lambda_m$  is a sequence of positive terms of increasing values, the value of the term  $(\lambda_m/\beta)^4$  is rapidly going to infinity, especially for small values of  $\beta$ . Therefore, the smaller the  $\beta$  (or, equivalently, the more rapidly  $A_{nm}$  values are going to zero, for a fixed  $n$ ), the smaller the required number  $M$ .

3.2 *The semiinfinite beam*

We consider next the problem of the flexure of a semiinfinite beam resting on a Winkler foundation of the hyperbolic type. We assume that the beam is subjected to a set of inhomogeneous boundary conditions. This means that at least one of the boundary conditions applied at the edge  $\xi = 0$  of the beam is an inhomogeneous boundary condition (at  $\xi = \infty$ , only homogeneous boundary conditions may be considered). Under these circumstances, the case of a beam subjected to a set of homogeneous boundary conditions may be considered as a special case.

As in the case of the finite beam, we require from the solution of (12a)  $w_0(\xi)$  to satisfy the set of the inhomogeneous boundary conditions of the problem, whereas from the solution of each one of (12b) we need  $w_n(\xi)$  ( $n = 1, 2, \dots$ ) to satisfy the set of the corresponding homogeneous boundary conditions. For the solution of the differential equations (12), a technique based on Laplace transforms will be outlined. To this end, we shall use the notation

$$\mathcal{L}[f(\xi)] = \int_0^\infty f(\xi) e^{-s\xi} d\xi = F(s) \tag{26}$$

to denote the Laplace transformation of a certain function  $f(\xi)$  (see, for instance, [10]).

Upon applying the Laplace transform on both sides of each one of (12), we obtain

$$w_0(\xi) = 4\mathcal{L}^{-1}[(s^4 + 4)^{-1}\mathcal{L}(p)] + \mathcal{L}^{-1}[(s^4 + 4)^{-1}B_0(s)], \quad n = 0 \tag{27a}$$

and

$$w_n(\xi) = \mathcal{L}^{-1} \left\{ (s^4 + 4)^{-1} \left[ 4\mathcal{L}(pw_{n-1}) - \sum_{i=0}^{n-1} \mathcal{L}(w_i w_n^{(iV)}) \right] \right\} + \mathcal{L}^{-1}[(s^4 + 4)^{-1}B_n(s)], \quad n = 1, 2, \dots \tag{27b}$$

where the operator  $\mathcal{L}^{-1}$  denotes the inverse of the Laplace transformation and

$$B_n(s) = s^3 w_n(0) + s^2 w'_n(0) + s w''_n(0) + w'''_n(0), \quad n = 0, 1, \dots \tag{28}$$

is a third-degree polynomial in  $s$ , whose constant coefficients depend on the boundary conditions applied at the end  $\xi = 0$ .

The inversion denoted in the second term of the right-hand side of (27a) and (27b) can be carried out analytically:

$$\begin{aligned} b_n(\xi) &= \mathcal{L}^{-1}[(s + 4)^{-1}B_n(s)] \\ &= w_n(0) \cos(\xi) \cosh(\xi) + \frac{1}{2} \left[ w'_n(0) + \frac{1}{2} w'''_n(0) \right] \sin(\xi) \cosh(\xi) \\ &\quad + \frac{1}{2} w''_n(0) \sin(\xi) \sinh(\xi) + \frac{1}{2} \left[ w'_n(0) - \frac{1}{2} w'''_n(0) \right] \cos(\xi) \sinh(\xi), \end{aligned} \tag{29}$$

$n = 0, 1, \dots$

On the other hand, whenever, in a particular problem, the external stress distribution  $p(\xi)$  is given in an analytical form, the inversions denoted in the first term of the right-hand side of eqns (27a) and (27b) may successively be carried out either analytically or numerically.

However, it must be noted that only two of the four boundary conditions appearing in (29) are known at the edge  $\xi = 0$  of the beam. The remaining two boundary conditions must be considered as two arbitrary constants. These constants will be determined so that regular behaviour of each one of the functions  $w_n(\xi)$  can be ensured at  $\xi = \infty$ .



Under these circumstances, it must be further noted that the solution technique outlined in this section might also be considered as an alternative technique for finite beam analysis. In that case, the two arbitrary constants entering the problem through each one of (29) must be determined so that each one of the functions  $w_n(\xi)$  satisfies the appropriate boundary conditions imposed at  $\xi = \beta = \lambda l$  (see also [11]).

### 3.3 The infinite beam

For analysis of an infinite beam resting on a linear elastic Winkler foundation and subjected to an external stress distribution  $p(\xi)$  symmetrical about the  $z$ -axis, an analytical technique based on Fourier integral representations has been developed in [1]. Here, the same technique, generalised for an arbitrary external stress distribution, will be employed to provide (12) with a recurrence-type solution.

The arbitrary stress distribution  $p(\xi)$  may be represented by a Fourier integral form (see, for instance, [10]), according to

$$p(\xi) = \int_0^\infty [p_0^*(\eta) \cos(\eta\xi) + q_0^*(\eta) \sin(\eta\xi)] d\eta, \tag{30a}$$

where

$$[p_0^*(\eta), q_0^*(\eta)] = \frac{1}{\pi} \int_{-\infty}^\infty p(\sigma) [\cos(\eta\sigma), \sin(\eta\sigma)] d\sigma. \tag{30b}$$

Hence, following the procedure outlined in [1], we obtain, for (12), a recurrence-type solution of the form

$$w_n(\xi) = \int_0^\infty \frac{p_n^*(\eta) \cos(\eta\xi) + q_n^*(\eta) \sin(\eta\xi)}{1 + (\eta/\sqrt{2})^4} d\eta, \quad n = 0, 1, \dots \tag{31}$$

where  $p_0^*(\eta)$  and  $q_0^*(\eta)$  are given by (30b) and

$$[p_n^*(\eta), q_n^*(\eta)] = \frac{1}{\pi} \int_{-\infty}^\infty \left\{ p(\sigma) w_{n-1}(\sigma) - \frac{1}{4} \sum_{i=0}^{n-1} w_i(\sigma) w_n^{(iV)}(\sigma) \right\} [\cos(\eta\sigma), \sin(\eta\sigma)] d\sigma, \quad n = 1, 2, \dots \tag{32a}$$

$$w_n^{(iV)}(\xi) = \int_0^\infty \frac{\eta^4 [p_n^*(\eta) \cos(\eta\xi) + q_n^*(\eta) \sin(\eta\xi)]}{1 + (\eta/\sqrt{2})^4} d\eta. \tag{32b}$$

Whenever the external stress distribution  $p(\xi)$  is given in an analytical form, each one of  $w_n(\xi)$  ( $n = 0, 1, 2, \dots$ ) may successively be evaluated, at any  $\xi \in (-\infty, \infty)$ , mostly numerically. Apparently, any one of the improper integrals appearing in (30)–(32) converges uniformly in  $(-\infty, \infty)$ , either because of the term  $(\eta/\sqrt{2})^4$  appearing in some of the denominators or because of the fact that usually the analytical form of  $p(\xi)$  is a decaying function of  $\xi$ . For these reasons, each one of  $w_n(\xi)$  ( $n = 0, 1, \dots$ ) is a decaying function of  $\xi$ .

## 4. APPLICATIONS

### 4.1 Finite beam subjected to a line load

As an application of the analysis developed in the preceding section, we consider here the problem of the flexure of a finite beam resting on a hyperbolic Winkler-type foundation. The beam is subjected to a line load  $P$ , with dimensions of force per unit length. The line load is applied normally at a point  $\xi = \xi_0$  of the beam. In this case, the external stress distribution  $p(\xi)$  may be expressed in an analytical form according

to

$$p(\xi) = \bar{P} \delta(\xi - \xi_0), \quad \bar{P} = \lambda P, \tag{33}$$

where  $\delta(\xi - \xi_0)$  is Dirac's delta function.

We further assume, for simplicity, that the beam is subjected to a set of homogeneous boundary conditions and let  $X_m$  be the set of corresponding characteristic beam functions. Then, (21) gives

$$p_m = \frac{\bar{P}}{\beta \|X_m\|^2} X_m \left( \frac{\lambda_m \xi_0}{\beta} \right), \quad m = 1, 2, \dots \tag{34}$$

Under these circumstances, the asymptotic expansion solution (13) may be obtained in the form

$$w(\xi) = \left( \frac{\bar{P}}{k} \right) \sum_{n=0}^{\infty} \left( \frac{\bar{P}\mu}{k} \right)^n \beta^{n-1} w_n(\xi). \tag{35}$$

The functions  $\tilde{w}_n(\xi)$  are given, in nondimensional form, as follows:

$$\tilde{w}_n(\xi) = \sum_{m=1}^{\infty} \tilde{A}_{mn} X_m \left( \frac{\lambda_m \xi}{\beta} \right), \quad n = 0, 1, 2, \dots \tag{36}$$

where

$$\tilde{A}_{0m} = \frac{1}{\|X_m\|^2 [1 + (\lambda_m/\sqrt{2}\beta)^4]} X_m \left( \frac{\lambda_m \xi_0}{\beta} \right)$$

and

$$\begin{aligned} \tilde{A}_{nm} = & \frac{1}{\|X_m\|^2 [1 + (\lambda_m/\sqrt{2}\beta)^4]} \sum_{r=1}^{\infty} \sum_{l=1}^{\infty} \left\{ \left[ 1 + \left( \frac{\lambda_l}{\sqrt{2}\beta} \right)^4 \right] \left[ \tilde{A}_{(n-1)r} \tilde{A}_{0l} \right. \right. \\ & \left. \left. - \left( \frac{\lambda_l}{\sqrt{2}\beta} \right)^4 \sum_{i=0}^{n-1} \tilde{A}_{ir} \tilde{A}_{(n-i-1)l} \right] \right\} I_{mrl}, \quad m = 1, 2, \dots \end{aligned} \tag{37}$$

are nondimensional constant coefficients.

A comparison of (10) and (35) gives

$$w_n(\xi) = \left( \frac{\bar{P}}{k} \right) \beta^{-(n+1)} \tilde{w}_n(\xi), \quad \epsilon = \frac{\bar{P}\mu}{k} \tag{38}$$

and therefore

$$c = \frac{\epsilon}{\mu} = \frac{\bar{P}}{k}. \tag{39}$$

As was mentioned previously, both parameters  $c$  and  $\epsilon$ , defined as above, arise naturally in the formulation of the considered problem. Hence, for sufficiently small values of the nondimensional parameter  $P\mu/k$ , we expect uniform convergence of the asymptotic expansion (35). Furthermore, according to the presence of the factor  $\beta^{n+1}$ , which divides the  $n$ th term of the series in (35), the longer the beam is, the wider is the interval of values of the parameter  $P\mu/k$ , in which the asymptotic expansion (35) converges uniformly.

Finally, it might be of interest to point out the symmetry of the function  $\bar{w}_0(\xi)$  with respect to the variables  $\xi$  and  $\xi_0$ . Since the function  $(P/k\beta)\bar{w}_0(\xi)$  represents the solution of the corresponding linear problem, formed by (11a) and (33), this symmetry expresses physically the well-known "reciprocal theorem of Betti"; mathematically, it is a consequence of the fact that the function  $(4\beta)^{-1}\bar{w}_0(\xi)$  is the Green's function of the differential equation (11a). This observation, which might be of particular interest in problems related to the static or dynamic behaviour of finite beams resting on *linearly elastic Winkler-type media*, has been also pointed out in the forced vibration problem of strings and beams subjected to a concentrated unit load that is time harmonic[12].

4.2 Finite beam subjected to an initial displacement

As a second application of the analysis presented in Section 3, we consider now the problem of the flexure of a finite beam resting on a Winkler foundation of the hyperbolic type and subjected to the following set of inhomogeneous boundary conditions:

$$w(0) = \delta, \quad w'(0) = w''(\beta) = w'''(\beta) = 0. \tag{40}$$

These are the boundary conditions of a beam whose one edge  $\xi = 0$  is subjected to an initial displacement  $\delta$ , while the rotation is prevented and the other edge of the beam  $\xi = \beta$  is free.

It is further assumed that there is no external loading applied on the beam. Hence,  $p(\xi) = 0$  and the differential equations (12) are simplified as follows:

$$\begin{aligned} w_0^{(IV)} + 4w_0 &= 0, & n = 0 \\ w_n^{(IV)} + 4w_n &= - \sum_{i=0}^{n-1} w_i w_{n-i}^{(IV)}, & n = 1, 2, \dots \end{aligned} \tag{41}$$

where, according to the solution procedure developed in Section 3.1, the unknown functions  $w_n(\xi)$  ( $n = 0, 1, 2, \dots$ ) must satisfy the following boundary conditions:

$$\begin{aligned} w_0(0) = \delta, \quad w_0'(0) = w_0''(\beta) = w_0'''(\beta) &= 0, & n = 0 \\ w_n(0) = w_n'(0) = w_n''(\beta) = w_n'''(\beta) &= 0, & n = 1, 2, \dots \end{aligned} \tag{42}$$

It is apparent that  $w_0(\xi)$  must satisfy the set of the inhomogeneous boundary conditions of the problem while each one of  $w_n(\xi)$  ( $n = 1, 2, \dots$ ) must satisfy the set of the corresponding homogeneous ones, namely, the boundary conditions of a clamped-free beam.

Under these circumstances, the asymptotic expansion solution (13) may be obtained in the form

$$w(\xi) = \delta \sum_{n=0}^{\infty} (\mu\delta)^n \bar{w}_n(\xi), \tag{43}$$

where the functions  $\bar{w}_n(\xi)$  are given, in nondimensional form, as follows:

$$\bar{w}_0(\xi) = e^{\xi}[\bar{C}_1 \cos(\xi) + \bar{C}_2 \sin(\xi)] + e^{-\xi}[\bar{C}_3 \cos(\xi) + \bar{C}_4 \sin(\xi)], \quad n = 0 \tag{44a}$$

$$\bar{w}_n(\xi) = \sum_{m=1}^{\infty} \bar{A}_{nm} X_m \left( \frac{\lambda_m \xi}{\beta} \right), \quad n = 1, 2, \dots \tag{44b}$$

The nondimensional constants  $C_1, \dots, C_4$  appearing in (44a) are such that  $\bar{w}_0(\xi)$

satisfies the following boundary conditions:

$$\bar{w}_0(0) = 1, \bar{w}'_0(0) = \bar{w}''_0(\beta) = \bar{w}'''_0(\beta) = 0; \tag{45}$$

the functions  $X_m$  appearing in (44b) represent the set of characteristic beam functions of a clamped-free beam, and

$$\bar{A}_{1m} = \frac{1}{1 + (\lambda_m/\sqrt{2}\beta)^4} I_{00m}, \quad n = 1 \tag{46a}$$

$$\bar{A}_{2m} = \frac{1}{1 + (\lambda_m/\sqrt{2}\beta)^4} \sum_{i=1}^{\infty} \bar{A}_{i1} \left[ 1 - \left( \frac{\lambda_i}{\sqrt{2}\beta} \right)^4 \right] I_{0im}, \quad n = 2 \tag{46b}$$

$$\begin{aligned} \bar{A}_{nm} = \frac{1}{1 + (\lambda_m/\sqrt{2}\beta)^4} \sum_{i=1}^{\infty} \left\{ \bar{A}_{i(n-1)l} \left[ 1 - \left( \frac{\lambda_i}{\sqrt{2}\beta} \right)^4 \right] I_{0im} \right. \\ \left. - \left( \frac{\lambda_1}{\sqrt{2}\beta} \right)^4 \sum_{r=1}^{\infty} \sum_{i=1}^{n-2} \bar{A}_{ir} \bar{A}_{(n-i-1)l} I_{mr} \right\}, \quad n = 3, 4, \dots \end{aligned} \tag{46c}$$

where  $I_{mrl}$  is given by (25) and

$$I_{00m} = \int_0^1 [\bar{w}_0(\beta\eta)]^2 X_m(\lambda_m\eta) \, d\eta, \tag{47a}$$

$$I_{0im} = \int_0^1 \bar{w}_0(\beta\eta) X_i(\lambda_i\eta) X(\lambda_m\eta) \, d\eta. \tag{47b}$$

A comparison of (10) and (43) gives

$$w_n(\xi) = \delta \bar{w}_n(\xi), \quad \epsilon = \mu\delta, \tag{48}$$

and therefore

$$c = \frac{\epsilon}{\mu} = \delta. \tag{49}$$

Again, both parameters  $c$  and  $\epsilon$ , defined as above, arose naturally in the formulation of the stated problem. Hence, for sufficiently small values of the nondimensional parameter  $\mu\delta$ , we expect the uniform convergence of the asymptotic expansion (43).

Finally, it would be of practical interest to point out that the solution procedure developed in this section is also applicable to the case of a finite beam, of relative length  $2\beta$ , having free ends and subjected to an initial displacement  $\delta$  at its middle point. In this particular case, the displacement  $w(\xi)$  must be symmetric with respect to the middle point of the beam. The second of the boundary conditions (40), related to zero slope, guarantees this symmetry, provided that the point  $\xi = 0$  is considered as the middle point of the beam; the edges of the beam are located at  $\xi = \pm\beta$ .

### 5. NUMERICAL RESULTS AND DISCUSSION

For a numerical example, we consider here the case of a finite beam, with free ends, resting on a Winkler foundation of the hyperbolic type and subjected to a point load applied at the middle point  $\xi = \beta/2$  of the beam.

Since, for a free-free beam, the even characteristic beam functions  $X_2, X_4, \dots$  are antisymmetric with respect to  $\xi = \beta/2$ , only the terms containing the symmetric, with respect to  $\xi = \beta/2$ , functions  $X_1, X_3, \dots$  contribute in the summations denoted in eqn (36). This fact can easily be checked through eqn (37) and is in accordance with the physical problem. Indeed, since the point load is applied at the middle point of the

Table 1. The influence of  $N$  and  $M$  on the value  $\{\bar{w}(\beta/2)\}$  of the nondimensional middle point deflection of a finite beam subjected to a concentrated line load applied at its middle point

$M$	$N$						
	1	2	3	4	5	6	7
1	0.3333	0.4444	0.4815	0.4938	0.4979	0.4993	0.4998
2	0.5269	0.7089	0.7589	0.7697	0.7720	0.7729	0.7734
3	0.5415	0.7257	0.7734	0.7842	0.7870	0.7880	0.7884
4	0.5439	0.7281	0.7757	0.7866	0.7893	0.7903	0.7907
5	0.5442	0.7284	0.7760	0.7868	0.7896	0.7906	0.7910

beam, the deflection curve of the beam, given by (35), must be symmetric with respect to  $\xi = \beta/2$ ; hence, each one of the functions  $w_n(\xi)$  ( $n = 0, 1, 2, \dots$ ) given by (36) must be also symmetric with respect to the middle point of the beam.

For numerical calculations, a CP6 Honeywell digital computer was used. The efficiency of the mathematical technique outlined in the previous sections can be ascertained by examining the numerical results illustrated in Table 1. There, for a beam foundation system with  $\beta = 3$  and  $\epsilon = (\bar{P}\mu/k) = 1$ , the values of the nondimensional middle point deflection  $\bar{w}(\beta/2) = (k/\bar{P})w(\beta/2)$  obtained by increasing the values of both numbers  $N$  and  $M$ , in which the series in (35) and (36), respectively, were truncated, are tabulated. The convergence of the presented results shows that five terms in the series (36) and seven terms in the series (35) were needed to provide results with an accuracy of three significant figures. However, as has already been mentioned, to obtain results of the same accuracy,  $M$  must be larger (smaller) for longer (shorter) beams, while  $N$  must be larger (smaller) for larger (smaller) values of  $\epsilon$ , provided that for those values of  $\epsilon$  the asymptotic expansion (35) still converges uniformly. It is also of interest to point out that the first column in Table 1 ( $N = 1$ ) shows the convergence of the corresponding numerical results in the case in which the beam is resting on a linear Winkler foundation ( $\epsilon = 0$ ).

The nondimensional deflection  $\bar{w}(\xi)$  of the left half of a beam, with  $\beta = 3$ , is illustrated in Fig. 2 for several values of  $\epsilon \in (0, 2)$ . The curve with  $\epsilon = 0$  represents the deflection curve when the beam is resting on a linear Winkler foundation; it gives results that are identical to those developed by Seely and Smith[13] (see also [1]), through the equations given by Hetényi[14]. In Fig. 3, the corresponding deflection curves have been plotted for an effectively longer beam,  $\beta = 5$ . From both Figs. 2 and 3, it is apparent that increasing the nonlinearity parameter  $\epsilon$  also increases the deflection of the beam.

Figure 4 illustrates the deflection  $w(\xi)$  of the left half of a free-free beam, with  $\beta$

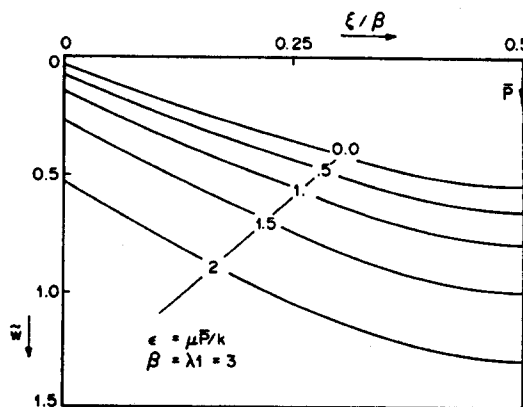


Fig. 2. Nondimensional deflection of the left half of a free-free beam resting on a Winkler foundation of the hyperbolic type and subjected to a concentrated line load applied at its middle point  $\{w(\xi) = (\bar{P}/k)\bar{w}(\xi)\}$ .

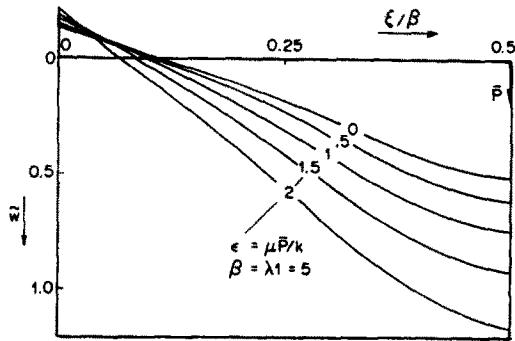


Fig. 3. Nondimensional deflection of the left half of a free-free beam resting on a Winkler foundation of the hyperbolic type and subjected to a concentrated line load applied at its middle point [ $w(\xi) = (P/k)\bar{w}(\xi)$ ].

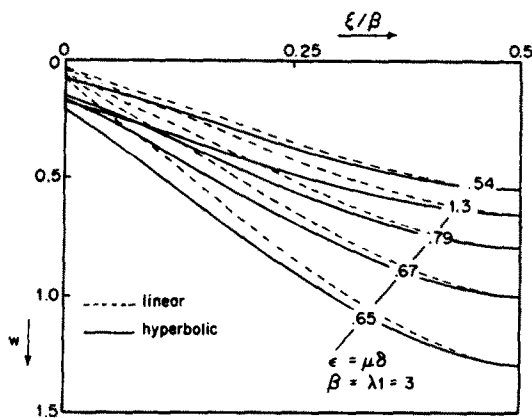


Fig. 4. Deflection of the left half of a free-free beam resting on a Winkler foundation of the hyperbolic type and subjected to an initial displacement at its middle point.

$= 3$ , which is subjected to an initial displacement at its middle point. The analysis presented in Section 4.2 is used to obtain these numerical results. Furthermore, for the purpose of comparison, attention was taken so that the middle point deflection of each one of the curves presented in Fig. 4 was the same as that of the corresponding curve illustrated in Fig. 2. To this end, the numerical value of the factor  $k/\bar{P}$  was considered as unity so that the value of the nondimensional parameter  $\epsilon$  in Fig. 2 would simply give the numerical value of the parameter  $\mu$  [units of  $(\text{length})^{-1}$ ]; this value multiplied by the numerical value of the middle point deflection of the corresponding curve then gives the value of the nondimensional parameter  $\epsilon = \mu\delta$  in Fig. 4. Hence, comparing corresponding curves, we can see that the effect of the initial displacement is more local than the corresponding effect of the applied load. This difference between corresponding curves becomes larger as long as we increase the value of  $\mu$ .

On the other hand, each one of the dashed lines showed in Fig. 4 represents the corresponding deflection curve of the beam when the Winkler foundation is a linear one ( $\mu = 0$ ). As in the case of the line load, it is again apparent that the effect of the nonlinearity parameter  $\epsilon$  is to increase the deflection of the beam.

## 6. CONCLUSIONS

This study presents the applications of a perturbation technique for the solution of the nonlinear equation governing the static problem of the flexure of a Bernoulli-Euler beam resting on a Winkler foundation of the hyperbolic type and subjected to an arbitrary external loading. Using this technique, the initially nonlinear problem has been reduced to the solution of linearised differential equations. For the successive solution of these linearised equations, some analytical methods have been outlined.

Among these analytical methods, of particular interest seems to be the one presented in Section 3.1, concerned with analysis of the problem of the flexure of finite beams. This method may easily be extended in two dimensions, so that the corresponding problem of homogeneous or cross-ply laminated thin elastic plates[15, 16], subjected to any set of homogeneous or inhomogeneous boundary conditions, can be solved. A further extension might also be made with respect to corresponding problems concerned with homogeneous or laminated composite thin elastic shell segments[17–19], provided that a suitable set of characteristic beam functions satisfying the shell edge boundary conditions may be found.

On the other hand, this method might also be used for the approximate solution of corresponding problems concerned with semiinfinite or infinite beams resting on a Winkler foundation of the hyperbolic type, especially when, from physical considerations, the deflection curve of the beam is expected to be a decaying function of the axial coordinate. In such a case, one has to replace the infinite beam with a finite one whose length  $l$  is large enough so that at locations remote from the origin the regular behaviour of the deflection curve can be ensured.

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APPENDIX

*On the completeness of the characteristic beam functions*

Let  $X_m(\lambda_m \xi / \beta)$  ( $m = 3, 4, \dots$ ) be the set of the characteristic functions for a free-free beam[7–9]. Each one of these functions, except the orthogonality conditions (15) and the property (16), is also satisfying the boundary conditions

$$X_m'' |_{\xi=0} = X_m'' |_{\xi=\beta} = X_m''' |_{\xi=0} = X_m''' |_{\xi=\beta} = 0, \quad m = 1, 2, \dots \tag{A1}$$

The functions

$$X_1 = c, \quad X_2 = c_1 + c_2 \xi \tag{A2}$$

also satisfy the boundary conditions (A1). Furthermore,

$$\int_0^\beta X_1 X_m d\xi = c \left( \frac{\beta}{\lambda_m} \right)^4 \int_0^\beta X_m^{(IV)} d\xi = c \left( \frac{\beta}{\lambda_m} \right)^3 X_m''' \Big|_0^\beta = 0, \quad m = 3, 4, \dots \tag{A3}$$

and

$$\begin{aligned} \int_0^{\beta} X_2 X_m d\xi &= c_2 \int_0^{\beta} \xi X_m d\xi = c_2 \left(\frac{\beta}{\lambda_m}\right)^4 \int_0^{\beta} \xi X_m^{(IV)} d\xi \\ &= c_2 \left(\frac{\beta}{\lambda_m}\right)^2 \left[ \left(\frac{\beta}{\lambda_m}\right) \xi X_m''' - X_m'' \right] \Big|_0^{\beta} = 0, \quad m = 3, 4, \dots \end{aligned} \quad (A4)$$

Equations (A2) and (A3) state that, in the interval  $(0, \beta)$ ,  $X_1$  and  $X_2$  are orthogonal to  $X_m$  ( $m = 3, 4, \dots$ ). Therefore, for the construction of a complete orthonormal basis of functions  $\Phi_m$  [see also (17)], the functions  $X_1$  and  $X_2$  must complete the set of the characteristic functions  $X_m$  ( $m = 3, 4, \dots$ ). To this end, and in order for the three unknown constants  $c$ ,  $c_1$  and  $c_2$  to be determined, the three orthogonality conditions,

$$\int_0^{\beta} X_i X_j d\xi = \beta \delta_{ij}, \quad i, j = 1, 2 \quad (A5)$$

must be used. Then, it can be shown that

$$X_1 = 1, \quad X_2 = \sqrt{3} \left(1 - \frac{2\xi}{\beta}\right). \quad (A6)$$

By following a similar procedure, it can be shown that the constant function

$$X_1 = 1 \quad (A7)$$

(in the case of a free-sliding or a sliding-sliding beam) or the function

$$X_1 = \sqrt{3} \left(1 - \frac{\xi}{\beta}\right) \quad (A8)$$

(in the case of a free-pinned beam) must be added to the set of the corresponding characteristic beam functions. Finally, introduction of the parameter  $\lambda_1 = 0$  (as well as  $\lambda_2 = 0$  in the case of the characteristic beam functions of the free-free beam) guarantees that for each one of these additional functions, the property (16) is also valid.